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Exploring Pre-service Teachers' Mental Constructions When Solving Problems Involving Infinite Sets

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ABSTRACT In this study, the researchers explore the meaning of the, *is equal to* symbol by exposing pre-service teachers (n = 38) to two tasks. The first task was_baseequivalence of two countable infinite sets. The second task focused on the comparison of the two numbers $0,\bar{9}$ and 1.The study was carried out at a South African university, where activity sheets comprising the two tasks were completed by pre-service teachers specializing to become high school mathematics teachers. Their written responses were analyzed against the initial itemized genetic decompositions formulated in accordance with APOS (Action-Process-Object-Schema) Theory. The findings of the exploration led to modified itemized genetic decompositions. These modified genetic decompositions have pedagogical implications. They also contribute to the APOS theory.

INTRODUCTION

In the Revised National Curriculum Statement, for grades R-12 (Schools) (DoE 2003) it is emphasized that curriculum and teacher development theories in recent times should focus on the role of teachers and specialists in the development and implementation of effective teaching. There is a need for teachers to know the ideas with which students often have difficulty and also find ways to help bridge common misunderstandings. For example, one important facet of mathematical knowledge is the ability to move flexibly among different representations (for example graphical, symbolic and verbal forms) of a mathematical concept. Teachers need to be able to make flexible use of representations before they are able to create an environment that allows learners the freedom to engage in problem solving (Brijlall et al. 2012). In the mathematics community of practice there is consensus that the students' ways of thinking should be taken into account when planning instruction and that teachers should choose or design sequences of lessons for use and discussion in class (Tsamir 2003). The researchers thought that this should also be applicable to university lecturers in the higher education arena. This study aimed to explore what mental constructions are evoked when the pre-service teachers express their thoughts in writing when answering particular mathematical questions. It aimed to explore the learners' mental construction of knowledge in mathematics while answering questions based on infinite sets. This was achieved by exploring the research question,

How do learners construct mathematical knowledge when answering questions involving infinite sets and recurring decimal numbers?

To unpack this question the researchers asked the following sub-questions.

- What do pre-service teachers' written responses reveal about their understanding of infinite sets?
- How do the pre-service teachers' mental constructions of action, process, and object derived from their written responses coincide with the preliminary itemizedgenetic decompositions for the study?

Constructivism gives a perspective of how learners learn and this study was underpinned by APOS theory, which has a constructivist grounding (Ndlovu and Brijlall 2015). The focus of the study was not only to understand how learners construct knowledge but also to explore the cognitive structures involved in the construction of knowledge. This theory clearly describes the cognitive structures used by learners to construct knowledge, through action, process, object and schema, hence the acronym APOS. Using APOS Theory, the construction of knowledge when dealing with problems involving infinite sets and recurring decimals, were explored through identifying the relevant initial itemized genetic decompositions.

Related Literature Review

In South Africa it was found that mathematics teachers lack the appropriate depth in pedagogical content knowledge (Brijlall and Maharaj 2014). Bansilal et al. (2014) also concurred with this state of teacher knowledge in South Africa. The notion of the infinite always presents difficulty to mathematics students. Tsamir (2003) studied the different types of representations, which arise when comparing two infinite sets. She worked with prospective teachers. Those teachers were given a set of odd numbers and a set of natural numbers and asked whether the set of natural numbers was larger than, smaller than or equal to the other set. She found that many students explained that the set of odd numbers was smaller since they were a subset of the set of natural numbers. That was an Israeli study and so a similar question was asked in the South African context. In a Korean study, Choi and Do (2005) studied the equality of $0,\overline{9}$ and 1, where represents the recurring 9 in the decimal expansion. The researchers found that the fundamental problem about two numbers being equal was related to the mathematical ideas inherent in extensions of number systems. Equality cannot be understood properly unless it is interpreted in terms of underlying mathematical structures. The researchers hence have decided to extend these findings in this study by exploring the mental structures pre-service teachers develop when engaging with the equality of $0,\overline{9}$ and 1. It was hoped that understanding such mental structures would be linked with the relevant mathematical structures mentioned in the study by Choi and Do (2005). Brijlall et al. (2011) studied the equality of $0,\overline{9}$ and 1. In that South African study, the researchers explored the intervention of APOS designed activity sheets to aid the understanding of the equality of $0,\overline{9}$ and 1. They found that most students (even after engaging with the activities) did not believe that the quantities $0,\overline{9}$ and 1 are equal. The researchers in that study did not focus on the concept of infinite sets, which are studied in this study.

Conceptual Framework

In recent developments the focus regarding mathematics learning is on the mental processes that an individual employs to understand a learnt concept (Brijlall and Ndlovu 2013). A number of learning theories, such as Piaget's theory on constructivism, Vygotsky's theory on scaffolding, and Skinner's theory on behavioral learning, focus on learning. This study mainly engages with APOS theory, which is a framework for the process of learning mathematics that pertains specifically to learning more complex mathematical concepts (Weyer 2010).

APOS theory is premised on the hypothesis that mathematical knowledge consists of an individual's tendency to deal with perceived mathematical problem situations by constructing mental actions, processes, and objects and organizing them into schemas to make sense of the situations and solve the problems (Dubinsky and McDonald 2008). This theory builds on Piaget's notion of reflective abstraction. According to Dubinsky (1991), reflective abstraction refers to the construction of logical mathematical structures by an individual during the course of cognitive development. Piaget (1967) distinguished three types of abstraction, which are empirical abstraction, pseudo empirical and reflective abstraction. Cetin (2009) stated that in empirical abstraction the focus is on general characteristics of objects and in reflective abstraction the focus is on the actions or operations done by a subject on mental objects. So he further elaborated that action, process, object and schema are the mental structures that an individual builds by the mental mechanism of reflective abstraction. Therefore, APOS allows for the development of ways of thinking about how abstract mathematics can be assimilated and learned (Cooley et al. 2006). In looking at mathematics, this theory is very much applicable in understanding learners' learning of different concepts in calculus such as derivatives and optimization problems.

According to Dubinsky (1991), there are five kinds of reflective abstractions: interiorization, encapsulation, coordination, reversal and generalization. These can be linked to the four stages of APOS. The descriptions of action, process, object and schema that follow are based on those given by Weller et al. (2009), Maharaj (2010, 2013) and Ndlovu and Brijlall (2015). A transformation is first conceived as an *action*, when it is a reaction to stimuli, which an individual perceives as external. It requires specific instructions, and the need to perform each step of the transformation explicitly. As an individual repeats and reflects on an action, it may be *interiorized* into a mental *process*. A process is a mental structure that performs the same operation as the action, but wholly in the mind of the individual. Specifically, the individual can imagine performing the transformation without having to execute each step explicitly. If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then one says the individual has *encapsulated* the process into a cognitive *object*. A mathematical topic often involves many actions, processes and objects that need to be organized and linked into a coherent framework, called a *schema*. It is coherent in that it provides an individual with a way of deciding, when presented with a particular mathematical situation, whether the schema applies. In the schema stage the learner has a collection of actions, processes, objects and other schemas that the learner understands in relation to calculus. These notions helped Brijlall and Ndlovu (2013) devise a linear model of an *itemized genetic decomposition*.

Itemized Genetic Decomposition

The genetic epistemology of Jean Piaget is found to be useful for this study. At the center of Piaget's work is a fundamental cognitive process which he termed "equilibration" (Piaget 1967). Through an application of the model of equilibration to a series of written tasks the researchers are able to generate an account of the arrangements of component concepts and cognitive connections prerequisite to the acquisitions of a mathematical concept (Brijlall and Bansilal 2010). Those arrangements, which are called "genetic decomposition", do not necessarily represent how trained mathematicians understand the concepts, but it gives a likely pathway that could enable a learner to understand the concept in question. Brijlall and Ndlovu (2013) introduced the notion of *itemized genetic decomposition*. They defined an itemized genetic decomposition (IGD) as a genetic decomposition specific to a mathematics task an individual is confronted with. For example, they spoke of an IGD for rectangle area, which dealt with a specific problem requiring learners to find the maximum area of a given rectangle. The researchers extend this notion of IGD in this study for the two tasks outlined in the methodology section of this paper.

METHODOLOGY

For this study the researches used a qualitative methodological paradigm. This section is presented in fivesubsections which are: 1) the case study, 2) background of participants, 3) ethical issues, 4) validity and reliability and 5) the two tasks and their IGDs.

The Case Study

In any type of dialogue it is effective when one uses a particular instance to illustrate something that is more general. It is easier to engage with your audience when you talk about real people and events instead of discussing theories and ideas that are abstract (Maree and Pieterson 2007). People generally understand an idea better if an example is used to illustrate the idea. The researchers are all familiar with specific details and that a single instance assists us to see how the abstract principles fit together (Maree and Pieterson 2007). "A case study is a specific instance that is frequently designed to illustrate a more general principle" (Nisbet and Watt as cited in Cohen et al. 2007: 253). The research was specific in that thirty-two pre-service teachers were invited from a South African university. The participants' written responses were coded as JT, the letters chosen from their names. For example, a participant with name John Thomas was coded JT. It was ensured that none of the codes were repeated.

Participants

The participants in our research were thirty-two pre-service teachers. The teachers were chosen for two reasons: firstly, the teachers were students from the fourth year class, which one of the researchers was teaching, and secondly, they were chosen purposively. Access is a key issue in research and it is a factor that must be considered early in the research procedure (Cohen et al. 2007:109). The participants were specializing in high school mathematics teaching. They were generally the better qualified (better matric entry mathematics mark) than the other students who were specializing to teach from grade1 to grade 9. The pre-service teachers who participated in this study had already passed five courses, which included differential and integral calculus, linear algebra, complex numbers and differential equations.

Ethical Issues

"A major ethical dilemma is that which requires researchers to strike a balance between the demands placed on them as professional scientists in pursuit of truth, and their subjects' rights and values potentially threatened by the research" (Cohen et al. 2007: 51). In terms of ethical considerations the researchers followed the procedures as stipulated by the university research office. Participation by those teachers was totally voluntary and their confidentiality, privacy and anonymity were assured. The consent letters included details of the study and data collection procedures. The participants were also assured that if they chose to be part of the study they could withdraw at any time without being prejudiced in any way.

Validity and Reliability

In qualitative research, "validity might be addressed through the honesty, depth, richness and scope of the data achieved, the participants, the extent of the triangulation and the disinterestedness or objectivity of the researcher" (Winter as cited in Cohen et al. 2007: 133). Validity can be improved through careful sampling, using the appropriate instruments and data analysis techniques (Henning 2004). Validity is not something that can be achieved absolutely but it can be maximized. According to Cohen et al. (2007: 149), reliability can be seen as the correlation between the researcher's recorded data and what actually happens in the natural setting of the research. This was achieved by analyzing the pre-service teachers' written responses against the itemized genetic decompositions formulated.

The Two Tasks and Their IGDs

The two tasks, which the pre-service teachers were asked to work on, follow next. The researchers also provide an IGD for each task.

Task One

The researchers thought that in order to correctly answer this question an individual need to possess a function schema, which incorporates the various APOS conceptions shown in Figure 1.

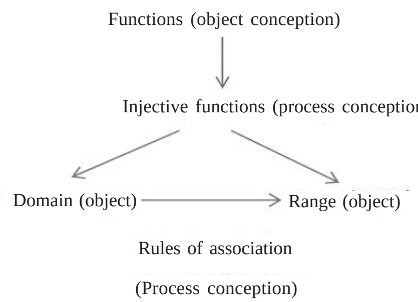


Fig. 1. An IGD for Task 1

In an IGD for task 1(which shall be referred to as IGD1), the researchers thought that the pre-service teachers were required to possess a function schema embedded in set theory. Within this schema, he/she should realize that a function is a set (of ordered pairs). The individual should also have an object conception of a function. Within his/her function schema, the individual would be expected to further classify a function a, b, ε D according to certain characteristics. For example a function $f: D \rightarrow R$ that satisfies the condition $f(a)=f(b) \Rightarrow a=b$ for all a,b,ε, D is an injective function. In this way the individual would be expected to have at least a process conception of an injective function. Such a process should entail an association between two sets, namely the domain and the range. For Task 1 if the individual could identify this domain as the set N and the set T as the range and also deduce an association (like, $n \rightarrow 2n-1$) then it can be proved that the elements of set N can be arranged in a one-to-one correspondence with the elements of the set T. Hence, the individual could conclude that those sets have the "same" number of elements.

Task 2

The question raised in Task 2 was:

Is $0,9\overline{9}=1$?

The researchers present two possible IGDs that could separately lead to an answer for this question.

An IGD they thought that could be useful is one based on a schema for infinite series as illustrated in Figure 2.This IGD shall be referred to as IGD2a henceforth.

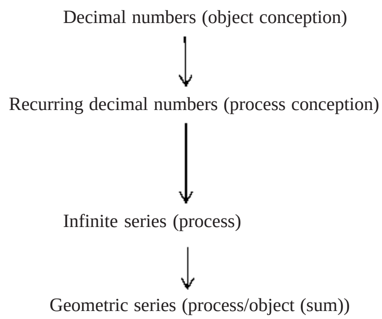


Fig. 2. An IGD (IGD2a) for Task 2

In IGD2a (see Fig. 2), an individual would require to have an object conception of decimal numbers. S/he would have encapsulated the mental manipulations of operations on decimal numbers and also identified the interrelatedness of decimal numbers within the real number (or complex number) system. If the individual has an object conception of decimal numbers, then one would expect such an individual to have at least a process conception of recurring decimal numbers. The individual should be able to identify recurring decimal numbers with the appropriate notation and should be able to disassemble a recurring number into sums. For example, $0,9\Sigma_{n=1}^{\infty}0,9\ (0,1)^{n-1}$ In this way the individual will have a process conception of infinite series. Within such a process conception of infinite series the individual would be expected to have a process conception of geometric series. It is denoted in Figure 2, two possible conceptions at the geometric series stage. One refers to a process conception of series and an object conception of the sum of a series. For example, it means that the individual has a process conception of an infinite series when working with 1+1/2+1/4+1/8+...and applying the operation of addition on an infinite number of real numbers. However, the individual is required to encapsulate such a process into an object conception when s/he acknowledges that such a sum equals 2 and hence sees1+1/2+1/4+1/8+... in its entirety as 2. In the context of 0,9 the representation $\Sigma_{n=1}^{\infty}\ 0,9\ (0,1)^{n-1}$ is at an object conception while the representation is at a process conception level.

Another possible way to solve this task would require an IGD referred to as IGD2b (see Fig. 3).The researchers thought that an individual would need to have an object conception of the system of rational numbers in place. Within this object notion of rational numbers s/he should disassemble this set into two categories of terminating decimals and non-terminating repeating ones. If the individual has a process conception at this stage it would suffice to attain success in solving task two. The individual further requires a process conception of conversion of decimals to rational numbers. When IGD2b was developed, findings of another South African research study by Conradie et al. (2009) were considered. They considered arguments offered by pre-service teachers on answering whether $0,\overline{9}=1$. The one argument is to let $x=0,\overline{9}$ which would imply that $10x=9,\overline{9}$.. By subtracting the first equation from the second one gets implying Hence, the agreement that0.9=1. This argument is based on interiorizing the conversion action into a process, which is represented

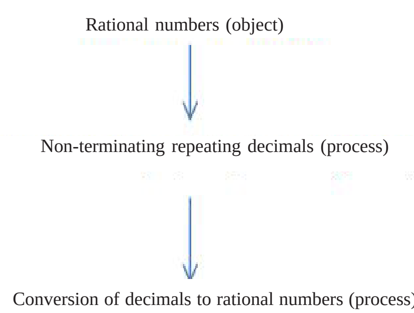


Fig. 3. An IGD (IGD2b) for Task 2

Another argument offered by some of the pre-service teachers in that study was to say that $0,\overline{3}=1/3$, so if one multiples both sides of the equation by 3 one will arrive at $0,\overline{9}=1$. Again such an action of operating on non-terminating repeating decimal numbers requires an individual to interiorize this into a process. Yet another argument was that $1/9=0,\overline{1}$, $2/9=0,\overline{2}$ $3/9=0,\overline{3}$ and so on, until one arrives at $8/9=0,\overline{8}$ and finally to $9/9,=0,\overline{9}$ or $1=0,\overline{9}$. Again one observes a process conception of recurring decimal numbers is necessary for the conceptualizing of this argument. These IGDs shall be considered when carrying out the discussion of the data in the next section.

RESULTS AND DISCUSSION

The researchers present and discuss the data for Task 1 and Task 2.

Task 1

For Task 1, where the sets of natural numbers and positive odd numbers were compared, Table 1 provides a summary of the written responses of the participants:

Table 1: The written responses to the "size" comparison of the two sets (n = 38)

Comparison category	*Number of responses in category*
doubtful	2
larger	15
smaller	11
equal	9
Not larger, smaller or equal	1

The majority of pre-service teachers (about 39%) thought that the set of natural numbers, *N*, had "more" elements than the set of positive odd numbers, *T*. Pre-service teacher SJ thought that this was the case since all the elements of set *T* are also found in set *N* (see Fig. 4). This explanation is a typical one in this category (larger) of participant responses. This would mean that this category of participant responses is working at the process level of understanding. They do not "see' these sets *N* and *T* as different objects and that appropriate processes, such as injective associations on these objects, are possible to elicit greater exploration to the problem in Task 1. It seems they believe that a subset of a set has to have "fewer" elements than the hereditary set and their conceptions are in the context of finite sets. This type of a perception also prevailed in an Israeli study (Tsamir 2003). Such a generalization is not applicable in the context of an infinite hereditary set and an infinite subset thereof.

Eleven (about 29%) of the pre-service teachers thought that the set *N* had fewer elements than the set *T* (see Table 1). A typical written response in this category (smaller) was that given by pre-service teacher TX (illustrated in Fig. 5). She worked with the elements of the given sets and considered their magnitudes. It seems that she had an inadequate process conception

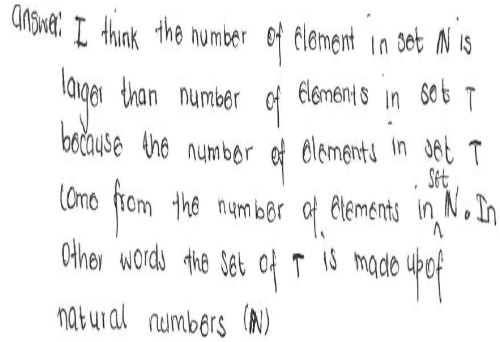

answer: I think the number of element in set N is
larger than number of elements in set T
because the number of elements in set T
come from the number of elements in set N. In
other words the set of T is made up of
natural numbers (N)

Fig. 4. The written response of SJ to Task 1

of the rule of association as indicated in IGD1 (see Fig. 1). She used the magnitude of some of the elements in set , for example 3, to incorrectly infer the 'size' of that set.

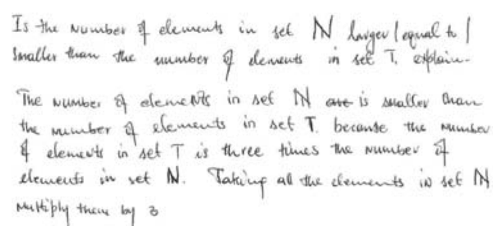

Is the number of elements in set N larger / equal to / smaller than the number of elements in set T, explain.
The number of elements in set N ~~are~~ is smaller than
the number of elements in set T because the number
of elements in set T is three times the number of
elements in set N. Taking all the elements in set N
multiply them by 3

Fig. 5. The written response of TX to Task 1

Note that Table 1 indicates that nine pre-service teachers (about 24%) responded that the sets *T* and *N* were equal. A typical written response for the category (equal) is given in Figure 6.

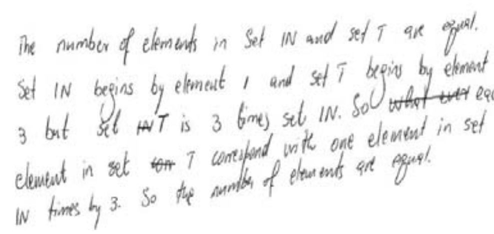

The number of elements in Set IN and set T are equal.
Set IN begins by element 1 and set T begins by element
3 but set ~~IN~~ T is 3 times set IN. So ~~what ever~~ each
element in set ~~IN~~ T correspond with one element in set
IN times by 3. So the number of elements are equal.

Fig. 6. The written response of MMP to Task 1

Pre-service teacher MMP concluded the sets as having "equal" elements on the basis of there being an injective association from the set *N* onto *T*. It seems that he has a process conception of the rule of association in IGD1 (see Fig. 1). His mistake though, was in describing the association from set T to N as $n \to 3_n$. This participant perceived the domain and range as objects. The responses in this category (equal) generally had explanations, which coincided with aspects of our initial itemized genetic decomposition in Figure 1. In the case of the two doubtful responses more than one description was provided. For example, pre-service teacher ZPM wrote larger and then later explains that it is smaller. For the category that had one response (not larger, smaller or equal) in Table 1, CNZ's argument is shown in Figure 7.

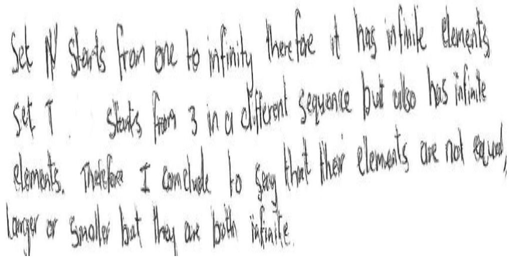


Fig. 7. The written response of CNZ to Task 1

This response (in Fig. 7) shows that pre-service teacher CNZ believed that one couldn't draw a conclusion regarding the 'size' of infinite sets. It seems that he did not find it possible to draw conclusions about equivalent infinite countable sets. This has implications for the researchers, in order to draw conclusions about the equivalence of infinite countable sets the researchers need to include these concepts in our initial itemized genetic decomposition, IGD1 (see Fig. 1). Also, mention of sequence is made in line two (see Fig.7) of his response. It means that the notion of infinite sequences, which Brijlall and Maharaj (2009) noted were dealt with in this module, needs to be considered in our modified itemized genetic decomposition for Task 1. Modified itemized genetic decompositions have proved to be useful as shown in the study by Ndlovu and Brijlall (2015).

Task 2

For Task 2 the researchers summarized the data in Table 2. The majority of the pre-service teachers (about 71%) indicated that they thought that the equality $0{,}\overline{9} = 1$ was invalid. This student perception was also found in a South African study by Brijlall et al. (2011). For the category 'yes', five of the responses were based on the fact that rounding off $0{,}\overline{9}$ made it equal to 1. So, although they made the correct choice, they had not provided suitable explanations. Pre-service teacher SJ managed to support his claim by providing a suitable argument (see Fig. 8).

Table 2: The written responses to the comparison of the two numbers

Category of response	*Number of responses*
Yes	7
No	27
Special cases	4

The argument by SJ is that if $0{,}\overline{9}$ is the closest number to one, it has to be 1. Her explanation is further substantiated by providing a conversion of the decimal expansion into 1. This indicated that there was a need to modify IGD2b (see Fig. 3). That modification appears in the conclusion of this paper.

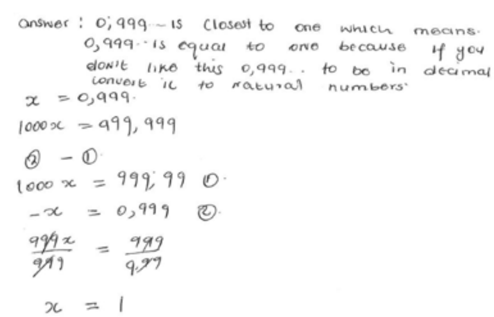
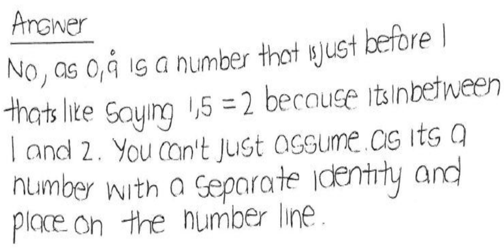


Fig. 8. The written response of SJ to Task 2

Pre-service teacher PC (see Fig. 9) responded that those two numbers could not be equal as they were completely different entities. In fact, she illustrates this difference by taking examples of unequal numbers to support her strong conviction that $0{,}\overline{9} \neq 1$. Tall and Schwarzenberger (1978) and Conradie et al. (2009), also found this type of reasoning in their study, where the students believed that $0{,}\underline{9} < 1$.

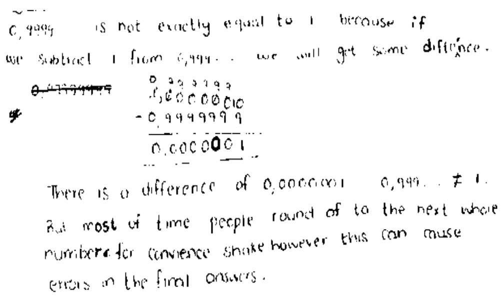
Pre-service teacher SS also thought that these numbers were different. Her argument is based on finding the difference between these two numbers. In Figure 10, the researchers ob-



Answer
No, as 0,9̊ is a number that is just before 1 thats like saying 1,5 = 2 because its inbetween 1 and 2. You can't just assume as its a number with a separate identity and place on the number line.

Fig. 9. The written response of PC to Task 2

serve that she had written $0,\overline{9}$ to nine decimal places and then found a non-zero difference.



0,999 is not exactly equal to 1 because if we subtract 1 from 0,999.. we will get some difference.

1,000000000
− 0,99999999
0,00000001

There is a difference of 0,0000001 0,999.. ≠ 1. But most of time people round of to the next whole numbers for convience shake however this can cause errors in the final answers.

Fig. 10. The written response of SS to Task 2

In this argument of SS it is observed that she possesses a process conception of a recurring decimal number. She has not encapsulated this process into an object conception. SS seemed to have a process conception of $0,\overline{9}$ and an object conception of the number 1. So this implied that the mental constructions for these two concepts were different and led to an incorrect conclusion, which SS made. This means that the inclusion of an object conception of recurring decimals in the modified genetic decomposition-needs to considered.

CONCLUSION

The researchers found that in general the pre-service teachers had difficulty in understanding the equivalence of two countable infinite sets, and also the comparison of the two numbers $0,\overline{9}$ and 1.With regard to the equivalence of the two countable infinite sets, their notions of finite sets guided the restricted conception of the participants. Regarding Task 2, the researchers found that generally participants had a process conception of $0,\overline{9}$ and this prevented them from concluding that the object $0,\overline{9}$ is equal to one. In other words they had a process conception for the left hand side representation and an object conception of the right hand side representation in the following equality:

$$0,\overline{9} = \sum_{n=1}^{\infty} 0,9\,(0,1)^{n-1}$$

This exploration was necessary as those pre-service teachers needed to be presented with a perspective of "knowing why" to the two fundamental problems they were exposed to, in the two tasks. This was found essential as those pre-service teachers needed to have a clear understanding of the concepts dealt with in order to confront the learners they would teach, once exiting the university.

RECOMMENDATIONS

There is a need for greater vigilance when dealing with extension of mathematical ideas related to the number systems and injective functions in the context of countable infinite sets. Further, the researchers recommend that a greater emphasis needs to be placed on the power of the '=' symbol in the context of identities, for example, $(a+b)^2 = a^2+2ab+b^2$, and equations in general.

The data analysis in this study led to modify both initial itemized genetic decompositions. Those modifications appear in Figures 11 and 12. The modified itemized genetic decompositions could inform future teaching in allowing students to make mathematically correct judgments on tasks such as those explored in this study. However, it should be noted that the results of this study should not be generalized, since this was a small case study.

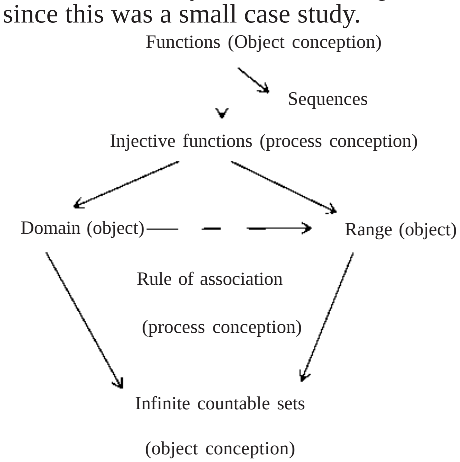


Fig. 11. A modified IGD for Task 1

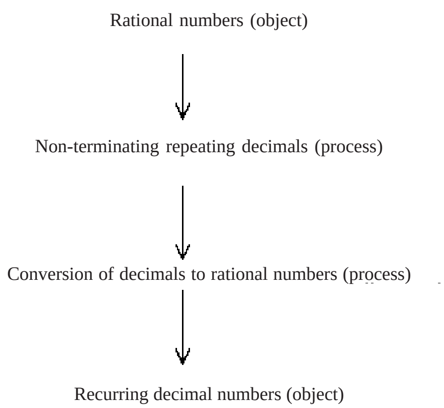


Fig. 12. A modified IGD for Task 2

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